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"An Extension of Maxwell's Electro-magnetic Theory of Light to include Dispersion, Metallic Reflection, and allied Phenomena." By EDWIN EDSER, A.R.C.S. Communicated by Captain W. DE W. ABNEY, C.B., F.R.S. Received February 18,—Read March 10, 1898.

The Electro-magnetic Theory of Light, as left by Maxwell, gave no explanation of dispersion, and led to conclusions in some respects inconsistent with the results of experiments on metallic reflection. There seems to be little doubt as to the general direction in which it would be necessary to modify that theory in order to give a satisfactory account of these phenomena. Electrical conduction has been considered by many to be inseparably connected with the motion of charged atoms, whilst the properties of a dielectric have been found to admit of an explanation on somewhat similar lines; consequently it would appear necessary, where fluctuations in the electric field of frequencies as great as those of light are concerned, to expressly formulate the reactions of the atoms or molecules composing the medium through which the disturbances are propagated. The mechanical theories of light, when modified in a similar manner, have been found capable of giving a more or less comprehensive account of dispersion and metallic reflection; and it would appear that the assumptions necessitated in the present case are at least as admissible as those which have been made elsewhere. No doubt a theory with any pretension to finality must include a satisfactory account of the nature of the luminiferous ether and of electricity, perhaps even of the ultimate constitution of atoms; and Mr. Larmor's investigations show how far we can even now go in this direction. On the other hand, a less comprehensive theory, depending only on the known laws of electrical actions, may prove not without value, if, whilst explaining the observed phenomena, it enables us to form a clear mental picture of the processes involved.



In what follows, I have endeavoured to extend Maxwell's theory so as to include dispersion and metallic reflection. The initial assumption that both conducting and dielectric media consist of molecules, each comprising (in the simplest case) two oppositely charged atoms, is essentially that made by Helmholtz in his paper on the "Electro-magnetic Theory of Dispersion."\* The methods I have employed are however different, and my final results, though bearing a general resemblance to those obtained by Helmholtz, differ from them in some important particulars, besides being simpler and more directly related to the results of investigations on other theories of the nature of light. But besides the gain in simplicity and consequent physical definiteness, I have had other reasons for not following Helmholtz's method. The utility of employing the principle of least action in this case may be questioned, since the only difficulty appears to lie in formulating the reactions due to the charged atoms, and the nature of these reactions is implied in the energy equations assumed. Moreover, the energy equations obtained by Helmholtz are themselves open to criticism, and some of the results obtained do not reduce to Maxwell's equations when the terms involving the polarization of the dielectric are equated to zero.†

2. This paper contains (1) a simple explanation of the fundamental phenomena observed in connection with a state of steady electrical strain in a dielectric; (2) a consideration of the law of propagation of electrical disturbances in a polarizable medium; equations are obtained which explain dispersion, both ordinary and anomalous; (3) and a consideration of metallic reflection, especially in connection with Kundt's experiments on the velocity of light in metals.

#### *State of Steady Electrical Strain in a Dielectric.*

3. It is here sought to explain Faraday's discovery, that the introduction of a material dielectric between the plates of a charged condenser, diminishes the potential difference between the plates.

\* H. von Helmholtz, 'Wied. Ann.', 1893, vol. 48, pp. 389–405, 723–725. Translated by Dr. Howard in the 'Electrician,' vol. 37, pp. 404–408.

† Helmholtz's Theory of Dispersion has been criticised by Rieff ('Wied. Ann.', 1895, vol. 55, pp. 82–94) and Heaviside ('Electrician,' vol. 37, Aug. 7, 1896). In addition it would appear that the employment of a term expressing the dissipation of energy (other than Rayleigh's "dissipation function") is inadmissible in the principle of least action (Larmor, 'Brit. Assoc. Report,' "On the Action of Magnetism on Light," 1893 (Nottingham)). The final equations obtained by Helmholtz, when the terms relating to the polarization of the medium are equated to zero, are of the nature

$$\dot{\mathbf{B}} = \text{curl } \mathbf{E} \text{ and } -4\pi\dot{\mathbf{D}} = \text{curl } \mathbf{H},$$

where  $\mathbf{B}$ ,  $\mathbf{E}$ ,  $\mathbf{D}$ , and  $\mathbf{H}$  represent the (vector) magnetic induction, electromotive intensity, displacement, and magnetic force respectively. They differ in sign from Maxwell's equations.

Let us consider a dielectric medium, other than the ether, to be composed of molecules each comprising, in the simplest case, two oppositely charged atoms at a definite distance apart. The volume actually contained by the atoms is provisionally assumed to be small in comparison with the volume of the interatomic spaces.

In what follows the term *axis of a molecule* will be applied to the vector distance between a negative and its associated positive atom. In an isotropic medium the axes of the various molecules will, in the absence of electrical strain, be inclined indifferently in all directions, so that any element of volume will have no resultant electric moment. If now a difference of potential be established between any two parallel planes in the medium, the positive atoms will all move toward points of lower, and the negative atoms toward points of higher potential. As a result each element of volume will now possess a resultant electric moment. Also, if we define a molecular electric moment as the product of the atomic charge (taken as positive) into the vector distance between a negative and its associated positive atom, the resultant electric moment due to a strained element of volume containing a number of molecules, will coincide in direction with the direction of fall of potential.

In the unstrained medium we may represent the several moments of the molecules contained in an element of volume by lines radiating uniformly from a point, and ending on the surface of a sphere.\*

To find the electric moment of an element of the strained medium, it is only necessary to determine the sum of the alterations of the component molecular moments in the direction of the fall of potential.

4. Let  $P$  be the electromotive intensity at a point in the dielectric, and let  $q$  be the atomic charge, the distance between two associated atoms being  $l$ . Then a molecule whose axis makes an angle  $\theta$  with the direction of fall of potential, will be subjected to a couple  $Pql \sin \theta$  tending to decrease  $\theta$  and each atom will be subjected to a force  $Pq \cos \theta$  tending to increase  $l$ . Assuming that the forces of restitution called into play by the displacement of the atoms are in both cases proportional to the linear displacements, we shall have

$$\frac{l}{2}\phi = K_1 Pq \sin \theta, \quad \frac{\Delta l}{2} = K_2 Pq \cos \theta,$$

where  $\phi$  indicates the (infinitesimal) molecular rotation, and  $\frac{1}{2}\Delta l$  is the linear displacement of either atom in a line with the axis of the molecule.

The increase of the component molecular moment in the direction of the fall of potential due to these strains, will be

$$ql\phi \sin \theta + q\Delta l \cos \theta.$$

\* Maxwell's 'Electricity and Magnetism,' vol. 2, § 443.

Substituting for  $\phi$  and  $\Delta l$  this expression becomes

$$2K_1Pq^2 \sin^2\theta + 2K_2Pq^2 \cos^2\theta.$$

Also, if  $dv$  = the volume of the element of the dielectric, and if there are  $n$  molecules per unit volume, the number of molecules whose axes are inclined at angles between  $\theta$  and  $\theta + d\theta$  will be

$$\frac{ndv}{2} \sin\theta \cdot d\theta.$$

Hence the resultant moment of the element  $dv$  of the strained medium will be

$$\begin{aligned} & ndvPq^2 \left\{ K_1 \int_0^\pi \sin^3\theta d\theta + K_2 \int_0^\pi \cos^2\theta \sin\theta d\theta \right\} \\ &= ndvPq^2 \left( \frac{4}{3}K_1 + \frac{2}{3}K_2 \right) \\ &= PMdv \dots \dots \dots \dots \dots \quad (1), \end{aligned}$$

$$\text{if } M = nq^2 \left( \frac{4}{3}K_1 + \frac{2}{3}K_2 \right) \dots \dots \dots \quad (2).$$

The electro-motive intensity, parallel to the direction of the fall of potential, due to this element of the strained medium, at a point distant  $r$  from it in a direction making an angle  $\theta$  with the direction of the resultant moment, will be

$$-PMdv \frac{(1 - 3 \cos^2\theta)}{r^3}.$$

Taking the point at which we wish to determine the electromotive intensity as origin, and the axis of  $x$  parallel to the direction of fall of potential, we shall have for the electromotive intensity due to a slab of the strained dielectric, perpendicular to  $x$  and at a distance  $x$  from the origin (if  $dx$  = the thickness of the slab, and  $h = r \sin\theta$ )

$$-PM \cdot 2\pi dx \int_0^\infty \left( \frac{1}{(x^2 + h^2)^{\frac{3}{2}}} - \frac{3x^2}{(x^2 + h^2)^{\frac{5}{2}}} \right) h dh = 0.$$

The only slab of the dielectric which contributes anything to the electro-motive intensity at the origin is that lying between the planes at  $-\frac{1}{2}dx$  and  $+\frac{1}{2}dx$ . The electro-motive intensity due to this will be

$$-PM \cdot 2\pi dx \int_0^\infty \frac{h dh}{h^2 + \left( \frac{dx}{2} \right)^{\frac{3}{2}}} = -4\pi PM.$$

Hence if  $D$  be the displacement other than that produced by the polarization of the medium

$$P = 4\pi D - 4\pi MP$$

$$\therefore P(1 + 4\pi M) = 4\pi D \dots \dots \dots \quad (3).$$

Hence  $(1 + 4\pi M)$  is the value of the specific inductive capacity measured in the electro-static system. In the electro-magnetic system (3) must be written

$$P\kappa \left( 1 + \frac{4\pi M}{\kappa} \right) = 4\pi D \dots \dots \dots \quad (4),$$

where  $\kappa$  is the dielectric constant of the ether.

5. A few remarks may be made in connection with the assumptions made in the course of the above argument. An objection might be raised to the supposition that two oppositely charged atoms could remain in equilibrium at a definite distance apart. To this it may be replied that the form of the argument does not necessitate the assumption that electrical forces are the only forces acting; and even if these were the only forces, it would be possible to account for the atomic separation by supposing both atoms to be revolving about their common centre of gravity.

If the atoms are considered to be small conducting spheres, and if the volume actually occupied by the atoms is a large fraction of the volume occupied by the dielectric, a correction on the lines of the Mosotti-Clausius dielectric theory would become necessary. On the other hand, if electrical conductivity is inseparably connected with the motion of charged atoms, it would appear to be wrong to consider the atoms themselves to be conductors.

If finite motions of the atoms were necessary to explain the properties of a dielectric, a treatment of the problem similar to that used by Weber, in his molecular theory of magnetism would be necessitated. It may, however, easily be shown that infinitesimal atomic displacements will account for the observed phenomena. Take the case of two plane and parallel charged condenser plates. Let  $P$  be the electro-motive intensity at a point between the plates when charged *in vacuo*. If now, the charge remaining the same, pure water is introduced between the plates, the electro-motive intensity will become

$$P - \frac{4\pi nq\delta}{\kappa},$$

if  $\delta$  = the average molecular displacement. Taking Cohn and Arons's value for the specific inductive capacity of water, viz., 76, we have

$$\frac{P}{P - \frac{4\pi nq\delta}{\kappa}} = 76.$$

Let  $P = 100$  volts. Then, since  $nq$  will equal the total charge carried by the oxygen in 1 gram of water, we shall have

$$\delta = \frac{75}{76} \cdot \frac{10^{10}}{12.5 \times \frac{10^4}{9} \times 9 \times 10^{20}} = 8 \times 10^{-16}.$$

Hence the magnitude of the displacement, necessary to account for the dielectric properties of a medium possessing the highest specific inductive capacity known, is small, even when compared with molecular magnitudes.

In determining the force acting on a charged atom within a polarised dielectric, a difficulty arises somewhat similar to that experienced when the force on a magnetic molecule within a magnet is sought.\* The above method, I think, is not open to very serious objection, and as the results obtained give a fair explanation of the experimental facts, it may, perhaps, be tentatively adopted.

[Added May 10.—Mr. Larmor† gives  $P + \frac{4}{3}\pi I$  as a first approximation to the value of the total electric field at a point within a polarized dielectric,  $I$  denoting the intensity of electrification, or the electric moment of the polarized medium per unit volume. As a consequence, the Lorenz refraction equivalent is obtained, whilst the relation  $(\mu^2 - 1)\infty$  density follows from the reasoning employed in the present paper. Further, a dispersion formula, differing somewhat from that of Ketteler, is obtained by Mr. Larmor; it would be interesting to know with what degree of accuracy this formula is capable of representing the dispersion of transparent substances, as determined, for instance, by Paschen and Rubens.‡

The following somewhat simple argument is submitted in justification of the assumption made above, that the field in a polarized medium is equal to  $P$ .

Let the space between two plane and parallel condenser plates, separated by a distance small in comparison with the area of either, be filled with an isotropic polarizable medium, such as that assumed above. The molecules are assumed to be in a state of equilibrium amongst themselves when the two plates are at the same potential; consequently the potential energy  $W$ , due to the small rotations  $\phi_1, \phi_2, \phi_3, \dots$  of the several molecules composing the medium may be represented by a quadratic function of the form

$$W = \frac{1}{2} \{ a_1 \phi_1^2 + a_2 \phi_2^2 + \dots + 2 b_{12} \phi_1 \phi_2 + 2 b_{13} \phi_1 \phi_3 + \dots \}.$$

A similar expression would obtain for the potential energy due to the separation of the atoms within the molecules. The conditions connecting the various coefficients are such that  $W$  is essentially positive.

Considering for a moment only the molecular rotations, the force

\* See Maxwell's 'Electricity,' vol. 1, p. 83, 1892 edition, footnote by Professor J. J. Thomson.

† "A Dynamical Theory of the Electric and Luminiferous Medium, Part III," 'Phil. Trans.,' A, vol. 190 (1897), § 18—21, pp. 232—236.

‡ Paschen, 'Wied. Ann.,' 1894, vol. 53, pp. 812—822; Rubens, 'Wied. Ann.,' 1895, vol. 54, pp. 476—485.

of restitution  $F_1$  called into play by the rotation  $\phi_1$  will be equal to  $dw/dh_1$ , where  $h_1 = \frac{l}{2}\phi_1$ . Thus

$$F_1 = \frac{2}{l} \{a_1\phi_1 + b_{12}\phi_2 + b_{13}\phi_3 + \dots\}.$$

The first term represents the force acting on an atom of the molecule whose subscript is unity, when that molecule alone is displaced; the succeeding terms represent the force on the same atom, due to the displacement of the remaining molecules throughout the medium. A number of relations, similar to those given by Maxwell,\* may be deduced, but require no further notice here.

Let the conducting surfaces, each of area  $A$ , and separated by a distance  $D$ , be charged by means of a battery to a constant potential difference  $V$ . Let  $\phi_1, \phi_2, \phi_3, \dots$  be the molecular rotations produced. If, when electrical equilibrium has been acquired, the molecular rotations are further increased by  $d\phi_1, d\phi_2, d\phi_3, \dots$ , the increment of electrical energy supplied by the battery will be equal to  $A \cdot VdI$ . Further, writing  $V = PD$ , we obtain for this energy  $E$ , the equivalent expressions

$$E = AD \cdot PdI_1 = P\Sigma lq \sin \theta d\phi,$$

where  $\Sigma$  indicates summation for the whole of the molecules throughout the medium. Since, on releasing the molecules, the above amount of electrical energy will be returned to the battery (the resistance of leads being considered negligible), we must have

$$P\Sigma lq \sin \theta_n d\phi_n = \Sigma 2d\phi_n \{a_n\phi_n + b_{n1}\phi_1 + b_{n2}\phi_2 + \dots\}.$$

Since the  $d\phi_i$ 's are arbitrary,

$$Plq \sin \theta_1 d\phi_1 = 2d\phi_1 \{a_1\phi_1 + b_{12}\phi_2 + b_{13}\phi_3 + \dots\}.$$

Hence  $F_1 = Pg \sin \theta_1$ . Consequently the couple acting on a polarized molecule will be equal to that due to a field  $P$ . Further, if  $P = 4\pi\sigma$ , it is easily seen that

$$\frac{2}{l} a_1\phi_1 = 4\pi\sigma \cdot q \sin \theta_1.$$

Similar reasoning will apply when the inter-atomic separations are considered. Consequently the total electrical field contributed by the molecules throughout the medium is numerically equal to  $4\pi I_1$ , and acts in a direction opposed to  $P$ , the result formerly obtained.]

\* 'Treatise,' vol. 1, chap. 3, §§ 87, 88.

## *Propagation of Electrical Disturbances.*

6. Only a very slight modification of Maxwell's equations appears to be necessary in order to determine the law governing the propagation of electrical disturbances in a polarized medium such as that previously considered.

Let  $P$ ,  $Q$ ,  $R$  be the components of the electromotive intensity at a point,  $F$ ,  $G$ ,  $H$  being the components of the vector potential there. The electromotive intensity due to the displacement of the atoms may, if the velocity of the atoms is small compared with the velocity of light, be derived from a potential  $\psi$ .

$$\left. \begin{aligned} P &= -\frac{dF}{dt} - \frac{d\psi}{dx} \\ Q &= -\frac{dG}{dt} - \frac{d\psi}{dy} \\ R &= -\frac{dH}{dt} - \frac{d\psi}{dz} \end{aligned} \right\} \dots \quad (5)$$

Here it must be remembered that  $\psi$  will be a function of  $t$ .

We may define the vector potential so that

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

In that case

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = -\nabla^2 \psi = 0 \quad \dots \dots \dots \quad (6),$$

since any element of volume, if taken so as to comprise a sufficient number of molecules, will contain as many positively charged as negatively charged atoms.

Let  $a, b, c$  represent the components of magnetic induction.

Then

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

with similar equations for  $b$  and  $c$ .

Hence, from (5),

$$\left. \begin{aligned} \frac{da}{dt} &= \frac{dQ}{dz} - \frac{dR}{dy} \\ \frac{db}{dt} &= \frac{dR}{dx} - \frac{dP}{dz} \\ \frac{dc}{dt} &= \frac{dP}{dy} - \frac{dQ}{dx} \end{aligned} \right\} \dots \quad (7)$$

These equations are identical with those used by Professor J. J. Thomson in his developments of Maxwell's theory.\* They express the law that however the electromotive intensity at any place may be modified by the presence of the charged atoms, its line integral round a closed curve will be equal to the rate of decrease of the induction through the curve.

These equations differ from those obtained by Helmholtz.<sup>†</sup>

7. If  $\alpha$ ,  $\beta$ ,  $\gamma$  are the components of the magnetic force, and  $u$ ,  $v$ ,  $w$  those of the electric current, then

$$\left. \begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz} \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx} \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy} \end{aligned} \right\} \dots \dots \dots \quad (8).$$

The current component  $u$  will include the total displacement current in the direction of  $x$ , i.e., not only that due to the variation of the induced electromotive intensity, but that also due to the continual alteration in position of the atoms throughout the medium. This will be equal to  $\kappa\dot{P}/4\pi$ . Mr. Heaviside<sup>†</sup> has further shown that in order to close the displacement current produced by a charge  $q$ , moving with a velocity  $v_1$ , we must add at the position occupied by  $q$  a current element such that its moment is  $qv_1$ . Hence in the present case the value of  $u$  will be given by  $\frac{\kappa\dot{P}}{4\pi} + \Sigma qv_x$ , where  $\Sigma qv_x$  indicates the summation, for unit volume, of the products of the several charges into their respective velocities parallel to the  $x$  axis.

Hence (8) may be written

$$\left. \begin{aligned} \kappa \dot{\mathbf{P}} + 4\pi \Sigma q v_x &= \frac{d\gamma}{dy} - \frac{d\beta}{dz} \\ \kappa \dot{\mathbf{Q}} + 4\pi \Sigma q v_y &= \frac{dx}{dz} - \frac{dy}{dx} \\ \kappa \dot{\mathbf{R}} + 4\pi \Sigma q v_z &= \frac{d\beta}{dx} - \frac{d\alpha}{dy} \end{aligned} \right\} \dots \dots \dots \quad (9)$$

\* 'Recent Researches in Electricity and Magnetism,' p. 252.

<sup>†</sup> The equations obtained by Helmholtz, besides differing in sign from the corresponding equations of Maxwell, would lead to the conclusion that the displacement produced by the charged atoms is circuital.

† O. Heaviside, 'Phil. Mag.', April, 1889, p. 324; see also J. J. Thomson, 'Phil. Mag.', April, 1881, July, 1889: also 'Recent Researches,' p. 19.

For a medium whose permeability is unity, we shall finally obtain, eliminating  $\alpha$ ,  $\beta$ ,  $\gamma$  between (9) and (7),

$$\left. \begin{aligned} \nabla^2 P &= \kappa \frac{d^2 P}{dt^2} + 4\pi \frac{d}{dt} \Sigma q v_x \\ \nabla^2 Q &= \kappa \frac{d^2 Q}{dt^2} + 4\pi \frac{d}{dt} \Sigma q v_y \\ \nabla^2 R &= \kappa \frac{d^2 R}{dt^2} + 4\pi \frac{d}{dt} \Sigma q v_z \end{aligned} \right\} \dots \quad (10)$$

8. Consider the propagation, parallel to the axis of  $z$ , of a train of plane waves, the electrical disturbances being parallel to the  $x$  axis. (10) reduces to

$$\frac{d^2P}{dz^2} = \kappa \frac{d^2P}{dt^2} + 4\pi \frac{d}{dt} \Sigma q v_x \dots \quad (11).$$

It only remains to express the last term on the right-hand side in terms of  $P$ .

Consider a molecule whose axis, as previously defined, makes an angle  $\theta$  with the axis of  $x$ . For simplicity, suppose both atoms comprised in the molecule to have equal masses. It has been shown by Mr. Heaviside and Professor J. J. Thomson,\* that a charged sphere moving with a velocity small in comparison with the velocity of light, has an apparent mass greater than its true mass by  $\frac{2}{3} \frac{q^2}{a}$ , where  $q$  is the charge, and  $a$  is the radius of the sphere. When other charged spheres are moving in the neighbourhood, a further correction might be necessary. Let, then,  $m$  represent the apparent mass of either atom in the molecule. The differential equation for the rotational displacement of the molecule will be of the form

$$2m\frac{l^2}{4}\frac{d^2\phi}{dt^2} + \frac{l^2}{2}\gamma\frac{d\phi}{dt} + \frac{1}{K_12}\phi - qlP\sin\theta = 0.$$

Here  $\gamma$  is the molecular viscosity, the other letters having the same meaning as in § 4.

If  $\gamma$  is so small that the time of free rotational vibration is not appreciably affected thereby, this equation may be written

$$\frac{d^2\phi}{dt^2} + \frac{\gamma}{m} \frac{d\phi}{dt} + \frac{4\pi^2}{\tau_1^2} \phi - \frac{2qP}{ml} \sin \theta = 0,$$

where  $\tau_1 = 2\pi\sqrt{K_1 m}$  = time occupied by a complete rotational vibration.

\* O. Heaviside, 'Phil. Mag.', April, 1889; 'Electrical Papers,' p. 505; J. J. Thomson, 'Recent Researches,' p. 21.

As we are here concerned only with forced vibrations, and therefore the particular integral of the above equation alone is required, we may write the solution to the above equation

$$\phi = \frac{2q \sin \theta}{ml} (D_1^{-1}P) \dots \dots \dots \quad (12),$$

where  $D_1^{-1}$  indicates the inverse of the operator  $D_1$ , and

$$D_1 = \frac{d^2}{dt^2} + \frac{\gamma}{m} \frac{d}{dt} + \frac{4\pi^2}{\tau_1^2}.$$

Similarly the equation to the atomic vibration along the axis of the molecule will be of the form

$$m \frac{d^2}{dt^2} \left( \frac{\Delta l}{2} \right) + \gamma \frac{d}{dt} \left( \frac{\Delta l}{2} \right) + \frac{1}{K_2} \frac{\Delta l}{2} - qP \cos \theta = 0.$$

The particular integral of this equation may be written

$$\Delta l = \frac{2q \cos \theta}{m} (D_2^{-1}P) \dots \dots \dots \quad (13),$$

where  $D_2^{-1}$  is the inverse of the operator  $D_2$ , and

$$D_2 = \frac{d^2}{dt^2} + \frac{\gamma}{m} \frac{d}{dt} + \frac{4\pi^2}{\tau_2^2}.$$

$$\tau_2 = 2\pi\sqrt{K_2 m}.$$

It has been assumed that the coefficient of viscosity is the same for both kinds of vibration.

Now the component of  $\Sigma p_{vx}$  contributed by the two atoms composing the molecule under consideration will obviously be obtained by differentiating with regard to time, the expression

$$ql\phi \sin \theta + q\Delta l \cos \theta.$$

Eliminating  $\phi$  and  $\Delta l$  by the aid of (12) and (13), this expression becomes

$$\frac{2q^2}{m} \{ \sin^2 \theta (D_1^{-1}P) + \cos^2 \theta (D_2^{-1}P) \}.$$

Hence employing precisely similar reasoning to that used in § 4, we finally determine that  $\Sigma q_{vx}$  will be obtained\* by differentiating with regard to time the expression

\* [Added June 13.—The possible presence of free ions, considered merely as isolated charged atoms, is not considered capable of materially affecting the dispersion formula for light waves. This would follow no less from theoretical considerations than from such facts as that the absorption of dilute sulphuric acid is not appreciably different from that of pure water. For very long electrical waves the case would be different.]

$$\begin{aligned} \frac{nq^2}{m} & \left\{ (D_1^{-1}P) \int_0^\pi \sin^2 \theta \, d\theta + (D_2^{-1}P) \int_0^\pi \cos^2 \theta \sin \theta \, d\theta \right\} \\ & = \frac{nq^2}{m} \left\{ \frac{4}{3}(D_1^{-1}P) + \frac{2}{3}(D_2^{-1}P) \right\}, \end{aligned}$$

$$\therefore \frac{d}{dt} \Sigma Q v_x = \frac{nq^2}{m} \frac{d^2}{dt^2} \left\{ \frac{4}{3}(D_1^{-1}P) + \frac{2}{3}(D_2^{-1}P) \right\}.$$

9. Substituting this value in (11), we obtain

$$\frac{d^2 P}{dz^2} - \kappa \frac{d^2 P}{dt^2} = \frac{4\pi nq^2}{m} \frac{d^2}{dt^2} \left\{ \frac{4}{3}(D_1^{-1}P) + \frac{2}{3}(D_2^{-1}P) \right\}.$$

Performing the operation denoted by  $D_1 D_2$  on the whole expression, we get

$$\left( \frac{d^2}{dz^2} - \kappa \frac{d^2}{dt^2} \right) (D_1 D_2) P = \frac{4\pi nq^2}{m} \frac{d^2}{dt^2} \left( \frac{4}{3} D_2 + \frac{2}{3} D_1 \right) P \dots \quad (14).$$

$$\text{Let } P = A_1 e^{\frac{2i\pi}{V\tau}(z-Vt)} \dots \dots \dots \quad (15).$$

Then since  $D_1 P = \left( -\frac{4\pi^2}{\tau^2} - i \frac{2\pi\gamma}{m\tau} + \frac{4\pi^2}{\tau_1^2} \right) P$ , with a similar expression for  $D_2 P$ , we obtain as the condition that (15) should be a solution of (14)

$$-\frac{4\pi^2}{V^2\tau^2} + \kappa \frac{4\pi^2}{\tau^2} = -\frac{4\pi nq^2}{m} \cdot \frac{4\pi^2}{\tau^2} \left( \frac{\frac{4}{3}}{\frac{4\pi^2}{\tau_1^2} - i \frac{2\pi\gamma}{m\tau} - \frac{4\pi^2}{\tau^2}} + \frac{\frac{2}{3}}{\frac{4\pi^2}{\tau_2^2} - i \frac{2\pi\gamma}{m\tau} - \frac{4\pi^2}{\tau^2}} \right).$$

Also  $\kappa = \frac{1}{V_0^2}$ , where  $V_0$  is the velocity of propagation of electrical disturbances in vacuo. Substituting this and simplifying we obtain

$$\frac{V_0^2}{V^2} = \mu^2 = 1 + \frac{c_1}{\frac{1}{\tau_1^2} - i \frac{\gamma}{2\pi m\tau} - \frac{1}{\tau^2}} + \frac{c_2}{\frac{1}{\tau_2^2} - i \frac{\gamma}{2\pi m\tau} - \frac{1}{\tau^2}} \dots \quad (16).$$

Here  $\mu$  is the refractive index of the medium, and

$$c_1 = \frac{\frac{4}{3}}{\frac{m\pi}{nq^2}} \frac{nq^2 V_0^2}{m\pi}, \quad c_2 = \frac{\frac{2}{3}}{\frac{m\pi}{nq^2}} \frac{nq^2 V_0^2}{m\pi}.$$

For transparent media,  $\gamma$  will be small, and the above equation may be written

$$\mu^2 = 1 + \frac{c_1}{\frac{1}{\tau_1^2} - \frac{1}{\tau^2}} + \frac{c_2}{\frac{1}{\tau_2^2} - \frac{1}{\tau^2}}.$$

Since both  $c_1$  and  $c_2$  are directly proportional to  $n$  the number of molecules per unit volume, it follows that in a medium which can be compressed without appreciably altering  $\tau_1$  and  $\tau_2$  we shall have

$$\mu^2 - 1 \propto \text{density}.$$

When  $\mu$  is very nearly equal to unity, this may be written

$$\mu - 1 \propto \text{density},$$

which is Gladstone and Dale's well-known law.

Further, the refractive index for infinitely long waves is obtained from the equation.

$$\begin{aligned}\mu_{\infty}^2 &= 1 + c_1\tau_1^2 + c_2\tau_2^2 \\ &= 1 + \frac{4}{3} \frac{nq^2V_0^2}{m\pi} \dots, 4\pi^2K_1m + \frac{2}{3} \frac{nq^2V_0^2}{m\pi} \cdot 4\pi^2K_2m \\ &= 1 + \frac{4\pi M}{K},\end{aligned}$$

which is the value obtained for the specific inductive capacity of a medium by (4).

Further, we may re-write (15),

$$\mu^2 = \mu_{\infty}^2 + \frac{c_1\tau_2^4}{\tau^2 - \tau_1^2} + \frac{c_2\tau_1^4}{\tau^2 - \tau_2^2}.$$

Also if  $V_0\tau_1 = \lambda_1$ ,  $V_0\tau_2 = \lambda_2$ ,  $V_0\tau = \lambda$ ,

this may be written

$$\mu^2 = \mu_{\infty}^2 + \frac{c'\lambda_1^2}{\lambda^2 - \lambda_1^2} + \frac{c''\lambda_2^2}{\lambda^2 - \lambda_2^2} \dots \quad (17),$$

where

$$c' = c_1\tau_1^2, \quad c'' = c_2\tau_2^2.$$

This is Ketteler's dispersion formula, which he has shown is capable of representing the optical properties of a large number of substances over a great range of values of  $\lambda$ . It is the same as that obtained by Mr. Glazebrook in his paper "On the Extension of Lord Kelvin's Contractile  $\mathcal{A}$ ether to include Dispersion, &c.,"<sup>\*</sup> and by a slight alteration in form reduces to Lord Kelvin's dispersion formula.<sup>†</sup> As its properties have been fully discussed in the papers referred to, nothing further need be said about it here.

The molecular constitution considered above is, of course, the simplest imaginable. With a more complicated molecule, since each

\* R. T. Glazebrook, 'Phil. Mag.', December, 1888.

† Lord Kelvin, Baltimore Addresses, 1884.

atom may have its own vibration periods, a much more complicated dispersion formula might be anticipated. In this connection the fact that the absorption spectra of the permanganates of potassium, sodium, lithium &c., are identical\* appears to be very suggestive.

For infinitely quick vibrations  $\mu$  becomes equal to unity. If the radiations discovered by Röntgen are ultimately proved to be due to transverse periodic disturbances of the ether, of very short wave length, this would explain why no refraction of these radiations is produced by material media.

10. The application of (16) to explain the optical properties of crystals and metals is obvious. In a crystal the co-efficients of the terms involving  $\tau_1$  and  $\tau_2$  will depend on the direction of vibration, which will result in  $\mu$  varying with the direction of propagation of light. Further, in accordance with a well-known mechanical principle  $\gamma$  must be taken into account, when  $\tau$  is nearly equal to  $\tau_1$  or  $\tau_2$ . In this case (16) may be reduced to the form

$$\mu^2 = R^2 (\cos 2\alpha + i \sin 2\alpha).$$

$\sin 2\alpha$  is essentially positive, whilst  $\cos 2\alpha$  may be either positive or negative. This is generally considered sufficient to explain the optical properties of metals, and of the quasi-metallic aniline dyes when in a solid state.

Kundt has pointed out that the velocity of red light in a metal is proportional to the electrical conductivity of that metal. A suggestive relation in this connection may be derived from (16).

Let  $2\alpha = \pi - 2\alpha'$ . Experiments with metallic films and prisms alike show that  $2\alpha'$  is small for the majority of metals. We may write

$$\mu = R(\cos \alpha' - i \sin \alpha').$$

The real part of the refractive index is therefore equal to  $R \sin \alpha'$ .

(16) may also be written in the form

$$\mu^2 = f(\tau) + i\gamma F(\tau).$$

Then, since  $\sin 2\alpha' = \tan 2\alpha' = \frac{\gamma F(\tau)}{-f(\tau)}$ ,

$$R \sin \alpha' = \frac{\gamma F(\tau)}{2\sqrt{(-f(\tau))}}, \text{ approximately.}$$

Mr. Tomlinson† has shown that in a number of cases the molecular viscosities of metals are in the same order of magnitude as their

\* Ostwald, 'Zeits. Phys. Chem.', vol. 9, p. 579.

† H. Tomlinson, 'Phil. Trans.', 1883, p. 168.

electrical resistances. Hence if  $\gamma$  in the present instance might be identified with the molecular viscosity as determined by Mr. Tomlinson, a connection similar to that derived experimentally by Kundt would be established. Calculating by the above approximate process the real part of the refractive index from Drude's experimental data, a fair agreement, as to order of magnitude, is found with the same quantity when calculated accurately. The agreement is not, however, close enough to explain the very accurate proportionality between the velocity of light in a metal and the conductivity of the latter which Kundt's figures imply. Since, however, Pflüger has shown that the temperature co-efficients of the velocities of light and the conductivities are not of the same order of magnitude, the process employed above may perhaps represent the nature of the physical connection between the two quantities to a sufficient degree of approximation.

11. [Added May 10.—Another relation of some importance can be readily obtained from (17). It is a well-established experimental law, often made the basis of exact chemical determinations, that the co-efficient of absorption of a solution of an absorbent substance in a transparent liquid is proportional to the number of molecules of the absorbent present in unit volume of the solution. A simple extension of the reasoning formerly used will give for the square of the refraction index of such a mixture the value

$$\mu^2 = 1 + \frac{c_1 n_1}{\frac{1}{\tau_1^2} - \frac{1}{\tau^2}} + \Sigma \frac{c_2 n_2}{\frac{1}{\tau_2^2} - \iota \frac{\gamma}{2\pi m \tau} - \frac{1}{\tau^2}} \dots \dots \dots \quad (18),$$

where  $c_1 n_1$ ,  $\tau_1$  are constants for the transparent medium, whilst  $c_2$ ,  $n_2$ ,  $\tau_2$ ,  $m$ , and  $\gamma$  refer to properties previously defined of the dissolved absorbent substance. Further we may write

$$\mu^2 = A + m_2 B,$$

where  $A$ , in dilute solutions, is nearly independent of the amount of colouring matter present, whilst

$$B = \frac{c_2 \frac{\gamma}{2\pi m \tau}}{\left( \frac{1}{\tau_2^2} - \frac{1}{\tau^2} \right)^2 + \frac{\gamma^2}{4\pi^2 m^2 \tau^2}} \dots \dots \dots \quad (19),$$

for a substance possessing only one absorption band, and is to a first approximation independent of the properties of the solvent.

Then

$$\mu^2 = R^2 (\cos 2x + \iota \sin 2x) = A + m_2 B.$$

When  $\alpha$  is small,  $\sin 2\alpha = \tan 2\alpha = \frac{n_2 B}{A}$ ,

$$R^2 = \sqrt{(A^2 + n_2^2 B^2)} = A, \text{ approximately.}$$

Hence

$$\mu = \sqrt{A + \epsilon \frac{n_2 B}{\sqrt{A}}}.$$

Here  $n_2 B / \sqrt{A}$  is the coefficient of absorption of the solution, and is proportional to  $n_2$ , the number of molecules of the absorbent substance present per unit volume. Since it is essential that (19) should possess a well-marked maximum value for a certain wave-length (that of the light absorbed), additional evidence is obtained in support of the introduction of a viscous term into the equations to the atomic vibrations.

The equation for the square of the refractive index of a substance like crystallized copper sulphate, which possesses marked selective absorption without exhibiting selective reflection to any extent, might be represented by an equation of the form of (18);  $\tau_2$ , &c., now referring to the conditions of motion of a particular atom, the motions of the remaining atoms giving rise to the terms involving  $\tau_1$ , &c.

“A Photographic Investigation of the Absorption Spectra of Chlorophyll and its Derivatives in the Violet and Ultra-violet Region of the Spectrum.” By C. A. SCHUNCK. Communicated by Dr. E. SCHUNCK, F.R.S. Received March 17,—Read March 24, 1898.

[PLATES 3, 4, 5.]

As is well known from the investigations of Soret,\* Gamgee,† and others, haemoglobin and its coloured derivatives show a characteristic absorption band lying between the lines G and M of the solar spectrum. The band has been shown to vary in position between narrow limits; in some derivatives it is nearer the red, and in others nearer the violet, end of the spectrum, and is of all the blood absorption bands the most stable.

The very near relationship that has been shown to exist by Schunck and Marchlewski‡ between phylloporphyrin (a chlorophyll derivative) and haematoporphyrin (a haemoglobin derivative), and the remarkable resemblance of their absorption spectra—one may almost

\* ‘Arch. des Sciences Phys. et Nat.,’ vol. 61, p. 322; vol. 66, p. 429.

† ‘Arch. des Sciences Phys. et Nat.,’ Dec., 1895.

‡ ‘Roy. Soc. Proc.,’ vol. 59, p. 233.